

# REMARKS ON SOME APPLICATIONS OF SKOROKHOD SPACE IN QUANTUM MECHANICS

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*Dedicated to A. V. Skorokhod*

**Abstract.** This paper discusses the role of the Skorokhod space and the convergence of probability measures on it in some recent studies of the foundations of quantum mechanics, both in the conventional setting over the real number field and in the more speculative one of nonarchimedean local fields.

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## 1. Introduction.

It is a pleasure and honour for me to have been asked to contribute to to the collection of articles marking the 40<sup>th</sup> anniversary of the discovery of the *Skorokhod topology* by Professor A. V. Skorokhod. I was a graduate student in Probability theory at the Indian Statistical Institute, Calcutta, India in 1956, and still remember vividly the surprise and excitement of myself and of my fellow students when the first papers on the subject by Skorokhod himself<sup>1</sup> and Kolmogorov<sup>2</sup> appeared. It was clear from the beginning that the space  $D$  with its Skorokhod topology would play a

fundamental role in all problems where limit theorems involving stochastic processes whose paths are not continuous (but are allowed to have only discontinuities of the first kind) were involved.

The present paper is a brief review of the use of the Skorokhod space and convergence of probability measures on it in some recent studies of quantum systems over fields and rings not only over the reals, but also over  $p$ -adic fields<sup>3,4</sup>. The first application I discuss is to the approximation of usual quantum systems by finite quantum systems<sup>3</sup>. The second<sup>4</sup> is a discussion of a path integral formalism applicable to a class of  $p$ -adic Schrödinger equations; the corresponding probability measure comes from a stochastic process with independent increments and is defined on the Skorokhod space of functions on  $[0, \infty)$  with values in a finite dimensional vector space over a nonarchimedean local field. This stochastic process and the associated measure therefore play the same role in the study of these  $p$ -adic Schrödinger equations as the brownian motion in the theory of the usual Schrödinger equations.

**2. Finite approximations of usual quantum systems.** The idea of studying finite quantum systems and their limiting forms goes back to Weyl<sup>5</sup> in the 1930's and Schwinger<sup>6</sup> in the 1960's, and has still retained great interest<sup>7</sup>. For both Schwinger and Weyl one of the themes was to approximate quantum systems over  $\mathbf{R}$  by finite quantum systems obtained by replacing  $\mathbf{R}$  with the cyclic group  $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$  for  $N$  large (this is also the basic idea in the so-called theory of the fast Fourier transform), identifying  $\mathbf{Z}_N$  with the grid  $\{0, \pm\varepsilon, \pm2\varepsilon, \dots, \pm k\varepsilon\}$  where  $N = 2k + 1$  and  $\varepsilon = (2\pi/N)^{1/2}$ . Weyl was interested only in the kinematics while Schwinger was interested in the dynamics also. Schwinger introduced the position coordinate  $q_N$  as the multiplication by the function  $k\varepsilon \mapsto k\varepsilon$  on the grid, and the momentum coordinate  $p_N$  as the *Fourier transform* of  $q_N$  on the finite group  $\mathbf{Z}_N$  using the identification above. Schwinger's principle was that the finite dimensional operator  $H_N^{(s)} = (1/2)p_N^2 + V(q_N)$  is a very good approximation to the energy operator  $H = (1/2)p^2 + V(q)$  for large  $N$ . Numerical work for the case of the harmonic oscillator showed that this was true<sup>3</sup>, and the question naturally arose if this could be substantiated by a limit theorem. In<sup>3</sup> it was shown that, in arbitrary dimension  $d$  and for potentials  $V$  which go to infinity faster than  $\log r$  at infinity on  $\mathbf{R}^d$ , we have

$$\|e^{-tH_N^{(s)}} - e^{-tH}\|_1 \longrightarrow 0 \quad (N \rightarrow \infty)$$

where  $\|\cdot\|_1$  is the trace norm (the condition on  $V$  insures that the operators  $e^{-tH}$  are of trace class for every  $t > 0$ ).

The method of proving this theorem is to use the Feynman-Kac formula<sup>8</sup> for the propagators of the Hamiltonian  $H$ . Such a formula is not available for the approximating Schwinger Hamiltonian  $H_N^{(s)}$ ; but, if one replaces the free Hamiltonian

by a second difference operator which is the discrete analogue of the Laplacian, then one has such a formula. One can call such Hamiltonians *stochastic* because the measure on the path space comes from a stochastic process with independent increments, namely the random walk, on the lattice  $\mathcal{L}_N = (\varepsilon \mathbf{Z}^d)$ . In the case of the finite approximation when the infinite lattice is truncated to a finite one, the path space measure still exists, but is now associated to a random walk with some boundary conditions that keep the walk inside the finite grid  $\mathcal{L}_N^*$ . It is not difficult to show that the Schwinger Hamiltonian is a better approximation than the stochastic Hamiltonian and so it is enough to establish the limit theorem for the stochastic ones. We shall denote these by  $H_N$  in the case of the infinite lattice and  $H_N^*$  in the case of the finite lattice.

For the continuum limit the path integral defining the propagator is with respect to the so-called brownian bridges, namely the measures  $\mathbf{P}_{x,y}^t$  defined by the conditional brownian motion starting from  $x$  at time 0 and exiting at time  $t$  through  $y$ . But, for the approximating processes, the measures are defined only on step functions with values in the lattices  $\mathcal{L}_N, \mathcal{L}_N^*$ . It is therefore essential, since one wants to discuss the approximation at the level of the probability measures on path spaces, to have all the measures defined on a single space. This has to be the Skorokhod space  $\mathcal{D}_t$  of functions on  $[0, t]$  with values in  $\mathbf{R}^d$  with discontinuities only of the first kind.

The fundamental result that allows one to prove the approximation theorem is the following local limit theorem on the Skorokhod space. Let  $\mathbf{P}_{N,a,b}^t$  be the conditional probability measure on  $\mathcal{D}_t$  for the random walk on the approximating lattice  $\mathcal{L}_N$  that starts from  $a \in \mathcal{L}_N$  at time 0 and exits from  $b \in \mathcal{L}_N$  at time  $t$ . Then

**Theorem** *Fix  $x, y \in \mathbf{R}^d$  and let  $a, b \in \mathcal{L}_N$  vary in such a manner that  $a \rightarrow x, b \rightarrow y$  as  $N \rightarrow \infty$ . Then*

$$\mathbf{P}_{N,a,b}^t \Longrightarrow \mathbf{P}_{x,y}^t$$

*in the sense of weak convergence of measures on  $\mathcal{D}_t$ .*

Let us now recall that the operators  $e^{-tH}$  and  $e^{-tH_N}$  are integral operators with kernels  $K_t, K_{N,t}$  where

$$K_t(x, y) = \int_{\mathcal{D}_t} e^{-\int_0^t V(\omega(s)) ds} d\mathbf{P}_{x,y}^t(\omega)$$

$$K_{N,t}(x, y) = \int_{\mathcal{D}_t} e^{-\int_0^t V(\omega(s)) ds} d\mathbf{P}_{N,x,y}^t(\omega)$$

The traces of these integral operators are calculated by integrating the kernels on

the diagonal. The theorem above now leads to the limit formula

$$\text{Tr}(e^{-tH_N}) = \sum_{a \in \mathcal{L}_N} K_{N,t}(a, a) \rightarrow \text{Tr}(e^{-tH}) = \int_{\mathbf{R}^d} K_t(x, x) dx$$

The second step is then to show that the trace limit relation continues to hold on going from  $\mathcal{L}_N$  to  $\mathcal{L}_N^*$ . This can be done, and one has the following limit formula:

$$\text{Tr}(e^{-tH_N^*}) = \sum_{a \in \mathcal{L}_N^*} K_{N,t}(a, a) \rightarrow \text{Tr}(e^{-tH}) = \int_{\mathbf{R}^d} K_t(x, x) dx$$

The required approximation of  $e^{-tH}$  by  $e^{-tH_N^*}$  in trace norm then follows from some standard arguments from functional analysis.

The limit theorem and its consequence require extensive use of techniques that are basic to the theory of the Skorokhod spaces and are discussed in detail in<sup>3</sup>.

### 3. $p$ -adic Schrödinger equations and path integral representations for their propagators in imaginary time.

Already in the 1970's and in fact much earlier even there was interest in understanding the structure of quantum mechanical theories over nonarchimedean local fields and even discrete structures like finite fields<sup>9</sup>. In recent years this interest has deepened, and mathematical and physical questions which may be viewed as the nonarchimedean counterparts of well-known quantum mechanical questions have begun to be studied over nonarchimedean fields<sup>10</sup>. In this section I shall discuss briefly one such aspect of  $p$ -adic analysis, namely, Schrödinger equations over  $p$ -adic fields; the proofs of the statements made here will appear elsewhere<sup>4</sup>.

Let  $K$  be any nonarchimedean local field of arbitrary characteristic and  $D$  a division algebra of finite dimension over  $K$ . We shall assume that  $K$  is the center of  $D$ ; this is no loss of generality since we may always replace  $K$  by the center of  $D$ . Let  $dx$  be a Haar measure on  $D$  and  $|\cdot|$  the usual modulus function on  $D$ :

$$d(ax) = |a|dx \quad (a \neq 0), \quad |0| = 0$$

It is then immediate that  $|\cdot|$  is a multiplicative norm which is ultrametric (i.e.,  $|x + y| \leq \max(|x|, |y|)$ ) that induces the original topology.

Let  $F$  be a left vector space of finite dimension over  $D$ . By a  $D$ -norm on  $F$  is meant a function  $|\cdot|$  from  $F$  to the nonnegative reals such that

- (i)  $|v| = 0$  if and only if  $v = 0$
- (ii)  $|av| = |a||v|$  for  $a \in D$  and  $v \in F$

(iii)  $|\cdot|$  satisfies the ultrametric inequality, i.e.,

$$|u + v| \leq \max(|u|, |v|) \quad (u, v \in F)$$

The norm on the dual  $F^*$  of  $F$  is a  $D$ -norm. If we identify  $F$  with  $D^n$  by choosing a basis, and define, for suitable constants  $a_i > 0$ ,

$$|v| = \max_{1 \leq i \leq n} (a_i |v_i|) \quad (v = (v_1, v_2, \dots, v_n))$$

it is immediate that  $|\cdot|$  is a  $D$ -norm. It is known that every  $D$ -norm is of this form. In particular all these norms induce the same locally compact topology on  $F$ .

For  $x \in F, \xi \in F^*$ , let us write  $x\xi$  for the value of  $\xi$  at  $x$ . If  $\chi$  is a nontrivial additive character on  $D$ , then  $\psi_\xi(x \mapsto \chi(x\xi))$  is an additive character of  $F$ , every additive character of  $F$  is of this form, and the map  $\xi \mapsto \psi_\xi$  is an isomorphism of topological groups from  $F^*$  to  $\hat{F}$ , the dual group of  $F$ . By  $\mathcal{S}(F)$  we denote the Schwartz-Bruhat space of complex-valued locally constant functions with compact supports on  $F$ . Let  $dx$  be a Haar measure on  $F$ . Then  $\mathcal{S}(F)$  is dense in  $L^2(F, dx)$ , and the Fourier transform  $\mathbf{F}$  is an isomorphism of  $\mathcal{S}(F)$  with  $\mathcal{S}(F^*)$ , defined by

$$\mathbf{F}(g)(\xi) = \int \chi(x\xi)g(x)dx \quad (\xi \in F^*)$$

For a unique choice of Haar measure on  $F^*$  we have, for all  $g \in \mathcal{S}(F)$ ,

$$g(x) = \int \chi(-x\xi)\mathbf{F}g(\xi)d\xi \quad (x \in F)$$

The measures  $dx$  and  $d\xi$  are then said to be *dual* to each other. For all of this, see<sup>11</sup>.

It is natural to call  $p$ -adic Schrödinger theory the study of the spectra and semigroups generated by operators in  $L^2(F)$  where  $F$  is a finite dimensional vector space over  $D$ , of the form

$$H = H_0 + V$$

Here  $H_0$  is a pseudodifferential operator and  $V$  is a multiplication operator. The simplest examples of  $H_0$  are as follows. We write  $M_b$  for multiplication by  $|x|^b$  ( $b > 0$ ) in

$$\mathbf{H} = L^2(F)$$

and, denoting by  $\mathbf{F}$  the Fourier transform on  $\mathbf{H}$ , we put

$$\Delta_{F,b} = \mathbf{F}M_b\mathbf{F}^{-1}$$

The Hamiltonian will then be of the form

$$H_{F,b} = \Delta_{F,b} + V$$

It is clear that over the field of real numbers and for  $b = 2$  the operator  $\Delta_{F,b}$  is just  $-\Delta$  where  $\Delta$  is the Laplace operator. The Hamiltonians  $H_{F,b}$  are thus the counterparts over  $D$  to the usual ones that appear in the conventional Schrödinger equations.

I shall now indicate how a path integral representation can be given for the propagators in imaginary time for the Hamiltonians  $H$  defined above in the nonarchimedean context. The key is the following.

**Proposition** *Fix  $t > 0$  and  $b > 0$  and let  $F$  be a  $n$ -dimensional left vector space over  $D$  with a  $D$ -norm  $|\cdot|$ . Then the function  $\varphi$  on  $V^*$  defined by*

$$\varphi(\xi) = \exp(-t|\xi|^b) \quad (\xi \in F^*)$$

*is in  $L^m(F, d\xi)$  for all  $m \geq 1$  and is positive definite. If we denote by  $f_{t,b}$  the (continuous) probability density on  $F$  whose Fourier transform is  $\varphi$ , then  $f_{t,b}$  is  $> 0$  everywhere. Moreover (i)  $0 < f_{t,b}(x) \leq f(0) \leq A t^{-n/b}$  for all  $t > 0$ ,  $A$  being a constant  $> 0$  not depending on  $t$  (ii) For  $0 \leq k < b$  we have, for all  $t > 0$  and a constant  $A > 0$  independent of  $t$ ,*

$$\int_F |x|^k f_{t,b}(x) dx \leq A t^{k/b}$$

It follows from this that the  $(f_{t,b})_{t>0}$  form a continuous convolution semigroup of probability measures which goes to the Dirac delta measure at 0 when  $t \rightarrow 0$ . Hence for any  $x \in F$  one can associate a separable  $F$ -valued stochastic process with independent increments  $(X(t))_{t \geq 0}$  with  $X(0) = x$ , such that  $f_{t,b}$  is the density of the distribution of  $X(t+u) - X(u)$  for any  $t > 0, u \geq 0$ .

Let  $D([0, \infty) : M)$  be the space of right continuous functions on  $[0, \infty)$  with values in the complete separable metric space  $M$  having only discontinuities of the first kind. For any  $T > 0$  we write  $D([0, T] : M)$  for the analogous space of right continuous functions on  $[0, T]$  with values in the complete separable metric space  $M$  having only discontinuities of the first kind, and left continuous at  $T$ . Then one can prove that the  $X$ -process has sample paths in the Skorokhod space  $D([0, \infty) : F)$ . More precisely we have .

**Theorem** *There are unique families of probability measures  $\mathbf{P}_x^b$  on  $D([0, \infty) : F)$  and  $\mathbf{P}_{x,y}^{T,b}(x, y \in F)$  on  $D([0, T] : F)$  , continuous with respect to  $(x, y)$ , such that*

$\mathbf{P}_x^b$  is the measure of the  $X$ -process that starts from  $x$  at time  $t = 0$ , and  $\mathbf{P}_{x,y}^{T,b}$  is the probability measure for the  $X$ -process that starts from  $x$  at time  $t = 0$  and is conditioned to pass through  $y$  at time  $t = T$ .

**Feynman–Kac propagator for  $e^{-tH_{F,b}}(t > 0)$**  From now on one can use standard arguments<sup>8</sup>, when  $V$  is bounded below and  $H_{F,b}$  is essentially self-adjoint on  $\mathcal{S}(F)$ , to show that the operator  $e^{-tH_{F,b}}(t > 0)$  is an integral operator in  $L^2(F)$  with kernel

$$K_{t,b}(x : y) \quad (x, y \in F)$$

which is represented by the following integral on the space  $\mathcal{D}_t = D([0, t] : F)$  :

$$K_{t,b}(x : y) = \int_{\mathcal{D}_t} \exp \left( - \int_0^t V(\omega(s)) ds \right) dP_{x,y}^{t,b}(\omega) \cdot f_{t,b}(x - y)$$

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